resit Test 2, NM2 (1)

Exc 1

$$A = \begin{bmatrix} -1 & 101 & & & & \\ -99 & -2 & 101 & & & \\ & -99 & -2 & 101 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -99 & -2 & 101 \\ & & & & -99 & -2 \end{bmatrix}$$

a) Show A irreducible

graph of A:

$$(1) \leftrightarrow (2) \leftrightarrow (3) \leftrightarrow \cdots \leftrightarrow (99) \leftrightarrow (100)$$

graph strongly connected

all nodes are connected to each other, path from node i to j and vice versa voor all i and j

$\Rightarrow$ A irreducible

b

$$A = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 100 & & & \\ -100 & 0 & 100 & & \\ & \ddots & \ddots & \ddots & \\ & & -100 & 0 & 100 \\ & & & -100 & 0 \end{bmatrix}$$

symmetric A_1 skew-symmetric A_2

$A = \frac{A^T + A}{2} + \frac{A^T - A}{2}$ (any other splitting in symm. and skew symm is fine as well)

0.1

0.1 A_1 symmetric $\Rightarrow$ $\lambda \in \sigma(A_1)$ real
A_1 irreducible

Gerschgorin on A_1:

'circles' $|z+2| \le 2$: $-4 \le z \le 0$ [−4 ——— −2 ——— 0]
'' real intervals $|z+1| \le 1$: $-2 \le z \le 0$ [−2 — −1 — 0]

0.4

$\cup$ 'circles' $= [-4, 0]$

since A_1 irreducible, boundary not included, since boundary not on every "circle" (Third Gerschgorin th.)

0.2 looking at columns gives similar result, since $A_1 = A_1^T$

$\Rightarrow$ $\sigma(A_1) \subset (-4, 0)$

0.1 A_2 skew-symmetric $\Rightarrow$ $\lambda \in \sigma(A_2)$ pure imaginary
A_2 irreducible

0.4 $\Rightarrow$ circles $|z| \le 100$ $\quad -100i \le z \le 100i$ $\quad \cup$ circles $= [-200i, 200i]$
intervals $|z| \le 200$ $\quad -200i \le z \le 200i$

0.2 A_2 irreducible, boundary not included, since boundary not on every circle

$\Rightarrow$ $\sigma(A_2) \subset (-200i, 200i)$

c. field of values $F(A) = \left\{ \frac{(x, Ax)}{(x,x)} \mid x \in \mathbb{C}^n, x \neq 0 \right\}$

- $\sigma(A) \subset F(A)$
- A normal, i.e. $AA^T = A^T A$ (if A real) $\Rightarrow F(A) =$ convex hull of $\sigma(A)$

0.2

- Bendixson: $F(A) \subset F\left(\frac{A+A^T}{2}\right) + F\left(\frac{A-A^T}{2}\right)$

$\Rightarrow \sigma(A) \subset F(A) \subset F\left(\frac{A+A^T}{2}\right) + F\left(\frac{A-A^T}{2}\right)$

$= \{ x + iy \mid x \in (0,4), y \in (-200i, 200i) \}$

↓

0.3 $\frac{A+A^T}{2}$ is normal because symm. $\Rightarrow F\left(\frac{A+A^T}{2}\right) =$ convex hull of $\sigma\left(\frac{A+A^T}{2}\right)$

$= (0, 4)$

$\frac{A-A^T}{2}$ is normal because skew symm. $\Rightarrow F\left(\frac{A-A^T}{2}\right) =$ convex hull of $\sigma\left(\frac{A-A^T}{2}\right)$

$= (-200i, 200i)$

Exc 2.

a. • $A^{(0)} = A$ QR method

0.4 $\left.\begin{array}{l} Q^{(i)} R^{(i)} = A^{(i)} \\ A^{(i+1)} = R^{(i)} Q^{(i)} \end{array}\right\}$ $i = 0, 1, 2, \ldots$

0.1 • $A^{(i)} \rightarrow$ upper triangular matrix for $i \to \infty$

• the eigenvalues on the diagonal of the converged $A^{(i)}$

0.2 are ordered from large to small in absolute value

• the convergence of element a_{kk} w.r.t. $a_{k-1,k-1}$

0.2 depends on $\left| \frac{\lambda_k}{\lambda_{k-1}} \right|$

assuming $|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_{n-1}| > \lambda_n$

b QR with shift

take shift s_i

in general $s_i = a_{nn}^{(i)}$

$\Rightarrow$ in step i use $A^{(i)} - s_i I$ instead of $A^{(i)}$

0.4 which has eigenvalues $\lambda_k - s_i$ $k = 1, \dots, n$

convergence speed in that step:

- order $|\lambda_k - s_i|$ from large to small, calling them $\mu_1, \dots, \mu_n$, then $|\mu_1| > |\mu_2| > \dots > |\mu_n|$

0.6 convergence for lower diagonal element; i.e $(A^{(i)} - s_i I)_{nn}$

$$\left| \frac{\mu_n}{\mu_{n-1}} \right|$$

once $A_{nn}^{(i)}$ converges to an eigenvalue, μ_n will become smaller and hence convergence gets fast, it becomes faster in each step

c determine Householder matrix H, s.t $H\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}*\\0\end{pmatrix}$

$H = I - 2vv^T$ when $\|v\| = 1$ $\left\{\text{since } \left\|\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}\right\| = 2 \Rightarrow H\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}\right\}$

take $\hat{v} := u - \|u\| e_1$ $u = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$ $\|u\| = \sqrt{1+3} = 2$

$\hat{v} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} - 2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix}$ $\|\hat{v}\| = \sqrt{1+3} = 2$

$v = \frac{\hat{v}}{\|\hat{v}\|} = \frac{1}{2}\begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix}$ v defines H: $H = I - 2vv^T$

1.2

$H = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - 2 \cdot \frac{1}{2}\begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix} \cdot \frac{1}{2}(-1 \quad -\sqrt{3})$

$H = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\\sqrt{3} & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\-\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}$

check $H\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}1/2 & -\frac{1}{2}\sqrt{3}\\-1/2\sqrt{3} & -1/2\end{pmatrix}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$ ✓

advantageous of using v instead of H

0.8
- v is vector, H is matrix $\Rightarrow$ v needs less memory/storage
- computing $x - 2vv^Tx$ is cheaper in computational effort than first computing H and than Hx

d) bring following matrix in Hessenberg form using a similarity transformation

$$A = \begin{bmatrix} -1 & 1 & \sqrt{5} \\ -1 & -2 & 6 \\ \sqrt{3} & -4 & -7 \end{bmatrix}$$

create a "zero" in position $A_{3,1}$

note $A(2:3,1) = \begin{pmatrix} -1 \\ +\sqrt{3} \end{pmatrix}$, with the H of part (c) this vector can be transformed to $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

0.9 ⇒ take transformation matrix $T = \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$

note $T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & H^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$

↑ H Householder; hence $H^T H = I$, $H^{-1} = H$

⇒ apply similarity transformation on A

$$T^{-1} A T = \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} -1 & 1 & \sqrt{5} \\ -1 & -2 & 6 \\ \sqrt{3} & -4 & -7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$$

$$\Rightarrow T^{-1} A T = \begin{bmatrix} -1 & 1 & \sqrt{5} \\ H\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} & H\begin{pmatrix} -2 \\ -4 \end{pmatrix} & H\begin{pmatrix} 6 \\ -7 \end{pmatrix} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$$

0.6

$$= \begin{bmatrix} -1 & 1 & \sqrt{5} \\ -2 & H\begin{pmatrix} -2 \\ -4 \end{pmatrix} & H\begin{pmatrix} 6 \\ -7 \end{pmatrix} \\ 0 & & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \begin{pmatrix} 1 & \sqrt{5} \\ H\begin{pmatrix} -2 \\ -4 \end{pmatrix} & H\begin{pmatrix} 6 \\ 7 \end{pmatrix} \end{pmatrix} H \\ -2 & \\ 0 & \end{bmatrix} = \begin{bmatrix} -1 & * & * \\ -2 & * & * \\ 0 & * & * \end{bmatrix}$$

Hessenberg form

e. first transforming A to Hessenberg form, i.e. $T^{-1}AT$ Hessenberg, explain why this is advantageous when applying QR method

1. Similarity transformation preserves the eigen-values

0.2

2. when starting with a matrix $\tilde{A}$ which is Hessenberg, the iteration $\tilde{A}^{(i+1)}$ in the QR

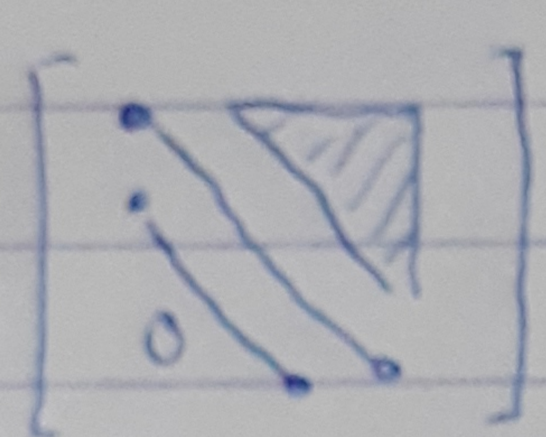
0.2 method are Hessenberg as well

3. QR method is essentially

$$\left.\begin{aligned} Q^{(i)} R^{(i)} &= \tilde{A}^{(i)} \\ \tilde{A}^{(i+1)} &= R^{(i)} Q^{(i)} \end{aligned}\right\} \quad i = 0, 1, 2, \ldots$$

making the QR factorization becomes simpler and computing RQ is simpler

Making QR:

$\tilde{A}^{(i)} =$ 

- only a 2×2 Householder matrix is needed to obtain a zero at position (2,1)

0.6 and only the first two rows need to be multiplied by the Householder matrix this is relatively cheap

- repeat for row 3 to make position (3,2) zero

  for row 4 to make position (4,3) zero

  etc.

Similar for computing RQ